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Abstract

This paper suggests an identification and estimation approach based on quantile regression to recover the underlying distribution of bidders' private values in ascending auctions under the IPV paradigm. The quantile regression approach provides a flexible and convenient parametrization of the private values distribution, with an estimation methodology easy to implement and with several specification tests. The quantile framework provides a new focus on the quantile level of the private values distribution, in particular the seller's optimal screening level, which can be very useful for bidders and seller. An empirical application using data from the USFS timber auctions illustrates the methodology.

JEL: C14, D44, L70

Keywords: Private values; timber auctions; ascending auctions; seller expected revenue; quantile regression identification; quantile regression estimation; quantile regression specification testing.

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1 Introduction

In an auction theoretical game, the bidders' private value cumulative distribution function (c.d.f.) is a key element for analysing the demand that a seller faces. Because it is required in the computation of the seller's expected payoff, its knowledge is crucial for policy recommendation, as e.g. the optimal reservation price policy. The issue here is that bidders' private values are not observed, thereby their distribution function is unknown for the econometricians and policy makers. The goal of this paper is to propose an identification and estimation approach based on quantile regression to recover the private value conditional distribution in ascending auctions.

In the past 20 years, several structural researches have proposed parametric and nonparametric approaches to identify and estimate the latent distribution of private values. A first wave of researchers has focused on parametric strategies. See Paarsch (1992) and Donald and Paarsch (1993, 1996) for Maximum Likelihood (ML) estimation, Laffont, Ossard and Vuong (1995) for simulated method of moments based on the revenue equivalence, Rezende (2008) for semiparametric linear regression models and Li and Zheng (2009) for a semiparametric Bayesian method in a model with endogenous entry and unobserved heterogeneity.

Nonparametric approaches for ascending auctions have been proposed in an attempt to circumvent the misspecification bias of parametric ones. See Haile and Tamer (2003) for independent private values (IPV) and Aradillas-Lopez, Gandhi and Quint (2013) for the affiliated setup. Menzel and Morganti (2013) built their nonparametric estimation on order statistics, which can be also viewed as sample quantiles. A few nonparametric approaches built on quantiles have been developed for first-price sealed-bid auctions. Using the insight in Haile, Hong and Shum (2003)[1], Marmer and Shneyerov (2012) argue that a quantile

[1]Haile et al. (2003) have shown that the quantiles of the private value distribution can be written in

approach makes the estimation of the private value p.d.f. easier from a derivation of the private value quantile function. On a different direction, Guerre and Sabbah (2012) propose to estimate quantile function instead of p.d.f..

Although nonparametric approaches have the advantages of being flexible in analysing the data at hand since no additional structure is imposed, it has some drawbacks as the curse of dimensionality and the need to choose for a smoothing parameter. The curse of dimensionality can be indeed a relevant estimation issue in view of important contributions to the empirical auction literature such as Haile and Tamer (2003) and Aradillas-Lopez et al. (2013), which consider, respectively, 5 and 6 explanatory variables for a sample size of a few thousands at best. By contrast, the quantile regression model suggested in this paper can be estimated with a parametric rate, independently of the dimension of the auction covariate, and does not involve the choice of a smoothing parameter.

The quantile regression model offers a flexible parametric specification for conditional quantiles since includes functional components that may be helpful to reduce the impact of misspecification. Compared to the semiparametric regression approach of Rezende (2008), quantile regression is computationally more difficult to perform but delivers an estimation of the full private value distribution, as needed for instance to derive an optimal reservation price. As a consequence, quantile regression is probably better suited for policy recommendations than a simpler regression approach. In particular, the quantile regression approach allows to highlight the screening level implied by the choice of a reservation price, a policy characteristic that has been mostly ignored by previous empirical approaches.

The identification strategy developed in this paper is built under the IPV setup and combines the nonparametric quantile approach in Marmer and Shneyerov (2012) and Guerre and Sabbah (2012) with a parametric quantile regression specification. The approach focus

terms of the quantiles, p.d.f. and c.d.f of bids on ascending auctions, but can be easily extended to other types of auctions (see Gimenes and Guerre (2014)). Ascending auctions are especially suitable for the identification of the private value distribution under the IPV paradigm because the transaction price equals the second-highest private value. As a result, the private value distribution can be nonparametrically identified through the winning bids distribution, as well known from Athey and Haile (2002)[2]. Based on data from the United States Forest Service (USFS) timber auctions, the empirical application suggests reservation price policies that may lead to a low probability of selling the good, especially when there is a sharp increase in the private value conditional quantiles. This is motivated by a high heterogeneity among the bidder, which may incentive sellers to screen bidders with low valuation from participating in the auction.

The paper is organized as follows. Section 2 describes the quantile and quantile regression identification approaches for ascending auctions. Section 3 provides the estimation methodology and asymptotic properties. Section 4 investigates the exclusion participation restriction. Section 5 provides an empirical application of the methodology. Finally, section 6 concludes the paper.

2 Identification

A single and indivisible object with characteristics $Z \in \mathbb{R}^d$ is auctioned to $N \geq 2$ bidders through an ascending auction. The seller sets a reservation price R prior to the auction that is the minimum price that he would be willing to accept. Both the set of auction covariates $X = (1, Z)$ and the number of actual bidders N participating in the auction are common knowledge. The object is sold to the highest bidder for the price of his last bid, provided that it is at least as high as the reservation price $R(X)$. Within the IPV paradigm, each

[2]See Haile and Tamer (2003) and Aradillas-Lopez et al. (2013) for extension.

bidder $i = 1, \ldots, N$ is assumed to have a private value V_i for the auctioned good, which is not observed by other bidders. The bidder only knows his own private value, but it is common knowledge for bidders and sellers that private values have been identically and independently drawn from a common c.d.f. $F_V(\cdot|X)$ conditional upon X, or equivalently, with a conditional quantile function $V(\alpha|X)$, $\alpha \in [0,1]$, defined as

$$V(\alpha|X) := \inf\{v : F_V(v|X) \geq \alpha\}. \tag{2.1}$$

When the private value conditional distribution is absolutely continuous with a probability density function (p.d.f) $f_V(\cdot|X)$ positive on its support $[V(0|X), V(1|X)] \subset \mathbb{R}_+$, as considered from now on, $V(\alpha|X)$ is the reciprocal function $F_V^{-1}(\alpha|X)$.

By the Fundamental Theorem of Simulation, $U_i = F_V(V_i|X)$, which can be viewed as the rank of a bidder with private value V_i in the population, is independent of X with a uniform distribution over $[0,1]$. The IPV paradigm implies that the ranks U_i, $i = 1, \ldots, N$, are independent. In other words, the dependence between the private values V_i and the auction covariates X can be fully captured by the nonseparable model

$$V_i = V(U_i|X), \quad U_i \overset{iid}{\sim} \mathcal{U}_{[0,1]} \perp X. \tag{2.2}$$

Therefore, bidders are identical up to the variable U_i, which represents the bidder ith's position in the private value distribution.

The quantile regression approach, developed by Koenker and Bassett (1978), restricts the quantile representation (2.2) to a regression specification, such as

$$\begin{aligned} V(\alpha|X) &= X\gamma(\alpha) \\ &= \gamma_0(\alpha) + Z\gamma_1(\alpha), \end{aligned} \tag{2.3}$$

where $\gamma_0(\alpha)$ is the quantile regression intercept and $\gamma_1(\alpha)$ the quantile regression slopes. Note that in (2.3), both the intercept and the slope quantile regression coefficients depend upon the rank α of the bidder in the population. Therefore, changes in the conditioning variables not only shift the location of the conditional distribution of V, but may also affect its scale and shape. A shock on the covariate X may affect a bidder with a low rank α in a different way than a bidder with a higher rank. A large discrepancy of the coefficients across α indicates high heterogeneity[3] among the bidders. As discussed later, taking into consideration such heterogeneity among the bidders may have important implications for both seller and bidders.

I now turn to the assumptions of the model. In the considered ascending auction, bidders raise continuously their prices and drop out of the auction as the prices reach their valuation.

Assumption 1 *The transaction price in an auction is the greater of the reservation price and the second-highest bidder's willingness to pay.*

Assumption 1 is an assumption on equilibrium play. This assumption was also used in Aradillas-Lopez et al. (2013) and, as noted in Athey and Haile (2002) and Bikhchandani, Haile and Riley (2002), is compatible with the multiple equilibria generated by ascending auctions. It is for instance the result of the dominant strategy equilibrium of a button auction, which is a stylized version of an ascending auction. Haile and Tamer (2003) use instead assumptions concerning bidder's behaviour, which determine the joint distribution of all bids.

The following three assumptions are required for the identification result.

[3]Note that heterogeneity and asymmetry are two concepts that should not be confused. Here, heterogeneity is concerned with the variation of $\gamma(\alpha)$ across quantile levels α, while asymmetry implies that different bidders can have different coefficients $\gamma(\alpha)$.

Assumption 2 *$V(\alpha|X)$ is strictly increasing and continuous on its support $[V(0|X),$ $V(1|X)]$ for all X.*

Assumption 3 *The private value conditional quantiles has a linear quantile regression specification*

$$V(\alpha|X) = X\gamma(\alpha). \tag{2.4}$$

Assumption 4 *The auction specific variable, Z, has dimension d, with a compact support in $\mathcal{Z} \subset (0, +\infty)^d$ and a nonempty interior.*

Assumption 2 and 3 deal with the quantiles of the bidders' private value distribution. Assumption 2 is usual in the quantile regression literature, whereas Assumption 3 imposes correct specification of the private value conditional quantiles. Assumption 4 concerns the auction specific covariate $X = (1, Z)$ and ensures that if $x\gamma_1 = x\gamma_2$, for all $x \in \mathcal{X} = \{1\} \times \mathcal{Z}$, thus $\gamma_1 = \gamma_2$.

Define

$$\Psi(t|N) = Nt^{N-1} - (N-1)t^N. \tag{2.5}$$

and let $W(\alpha|X, N)$ be the α-quantile of the winning bids conditional distribution given (X, N). It follows from Athey and Haile (2002, equation (5)) that $\Psi(F_V(\cdot|X)|N)$ is the distribution of the second-highest private value, which by Assumption 1 is equal to the winning bid. The next Lemma gives the cornerstone identification result of the quantile and quantile regression approaches.

Lemma 1 *Under IPV and assumptions 1 and 2, for each N and $\alpha \in [0, 1]$,*

1. *[**Probability of winning.**] A bidder with private value $V(\alpha|X)$ wins with probability α^{N-1};*

2. *[**Identification.**] The private value quantile maps to the winning bid quantile through*

$$V(\alpha|X) = W(\Psi(\alpha|N)|X, N), \tag{2.6}$$

where $\Psi(\alpha|N)$ is defined in (2.5).

3. *[**Stability property.**] If assumptions 3 and 4 also hold,*

   (i) There exists a vector of coefficients $\beta(\alpha|N)$ such that

   $$W(\alpha|X, N) = X\beta(\alpha|N);$$

   (ii) $\beta(\alpha|N)$ is uniquely defined and satisfies

   $$\beta(\Psi(\alpha|N)|N) = \gamma(\alpha). \tag{2.7}$$

Lemma 1-(1) shows that the rank α of a bidder in the population has a direct relationship with his probability of winning so that estimating $V(\alpha|X)$ can be helpful for newcomers that do not know the market or to benchmark a desired probability of winning the auction. Lemma 1-(2) gives the nonparametric identification of the private value conditional quantile through the observed winning bid conditional quantile function. Result (i) in Lemma 1-(3) is a stability property of the quantile regression specification, which is a consequence of Lemma 1-(2). Indeed, the latter shows that the winning bid quantile function admits the same linear specification as postulated for the private values, but for a transformed quantile level. Result (ii) in Lemma 1-(3) gives the identification result of the quantile regression approach. It shows that the coefficient $\gamma(\cdot)$ of the private value conditional quantile function is identified through the coefficient $\beta(\cdot|N)$ of the winning bid conditional quantile function, but evaluated at a different quantile level $\Psi(\cdot|N)$. The proof of Lemma 1 can be found in the supplemental material, which also groups the proof of all results stated in this paper.

2.1 Optimal Reservation Price

This section study the seller's expected payoff under a quantile perspective. Consider a binding reservation price set by the seller, i.e. $R(X) \in [V(0|X), V(1|X)]$. The reservation price thus plays the role of a screening level in the auction since bidders with $V(\alpha_i|X) < R(X)$ are prevented from participating in the game. Let $\alpha_R(X)$ be the screening level in the private value conditional distribution, i.e. $\alpha_R(X)$ is such that $R(X) = V(\alpha_R|X)$. It thus represents the percentage of bidders in the population that are not participating in the auction because of a low valuation. Note that the auctioned good will not be sold if all the players have valuation below $R(X)$, which implies that the probability of trading is $1 - \alpha_R(X)^N$. Therefore, for a given N, the probability of trading decreases with the screening level $\alpha_R(X)$.

Let the seller's payoff be defined as

$$\pi(r) = W\mathbb{I}(W \geq r) + V_0(1 - \mathbb{I}(W \geq r)), \tag{2.8}$$

where W is the winning bid, V_0 the seller's private value and $\mathbb{I}(A)$ an indicator function equal to 1 if the event A holds and 0 otherwise. The following proposition gives a quantile version for the seller's expected payoff, a candidate for the optimal screening level $\alpha_R^*(X, V_0) = \alpha_R^*(X, V_0(X))$ and for the corresponding optimal reservation price $R_*(X, V_0) = V(\alpha_R^*|X)$. Let $\Pi(\alpha|X, N, V_0)$ be the seller's expected payoff given (X, N, V_0) when the screening level is α.

Proposition 2 *Under IPV and assumptions 1-2,* *(i) The seller's expected payoff is, for a screening level α,*

$$\Pi\left(\alpha|X,N,V_0\right) = V_0\left(X\right)\alpha^N + R\left(X\right)N\alpha^{N-1}\left(1-\alpha\right) + N\left(N-1\right)\int_\alpha^1 V\left(t|X\right)t^{N-2}\left(1-t\right)dt, \tag{2.9}$$

where $V_0\left(X\right)$ is the seller's private value and $R\left(X\right)$ the reservation price;

(ii) The optimal reservation price $R_*\left(X,V_0\right) = V\left(\alpha_R^*|X\right)$ satisfies

$$R_*\left(X,V_0\right) - V^{(1)}\left(\alpha_R^*|X\right)\left(1-\alpha_R^*\left(X\right)\right) = V_0\left(X\right), \tag{2.10}$$

where $V^{(1)}\left(\alpha|X\right) = \partial V\left(\alpha|X\right)/\partial\alpha$ is the private value quantile density function.

Proposition 2 is a quantile version of a standard result in the auction literature, see Riley and Samuelson (1981, Proposition 1 and 3), Krishna (2010, p.23) and Myerson (1981). Note that since $V^{(1)}\left(\cdot|X\right) > 0$, it is clear that $\alpha_R^*\left(X\right) > \alpha_0\left(X\right)$, where $V_0\left(X\right) = V\left(\alpha_0|X\right)$. As well known, the optimal reservation price policy does not depend upon the number of bidders.

3 Estimation Methodology

Consider L i.i.d. ascending auctions $(W_\ell, Z_\ell, \ell = 1,\ldots,L)$. Let $X_\ell = (1, Z_\ell) \in \mathcal{X}$ be a row vector of dimension $d+1$. The following assumption concerns the variables in our model:

Assumption 5 *The variables $\{N_\ell, X_\ell, V_{i\ell},\ i = 1,2,\ldots N_\ell, \ell = 1,\ldots,L\}$ are independent and identically distributed. The support $\left[V\left(0|X_\ell\right), V\left(1|X_\ell\right)\right] \subset \mathbb{R}_+$ of $V_{i\ell}$ is bounded. Conditional on X_ℓ, the private values $V_{i\ell}$ are independent with common c.d.f. $F_V\left(\cdot|X_\ell\right)$ and a density function $f_V\left(\cdot|X_\ell\right)$ bounded away from zero.*

Assumption 5 implies that each auction is independent and that, within an auction, the IPV paradigm holds.

A standard approach in quantile regression interprets the coefficient $\beta(\alpha|N)$ in Lemma 1-(3) as the minimizer

$$\beta(\alpha|N) = \arg\min_{\beta\in\mathbb{R}^{d+1}} \mathbb{E}\left[\rho_\alpha (W - X\beta)|N\right],$$

where $\rho_\alpha(u) = u(\alpha - \mathbb{I}(u < 0))$. From (2.7), the unconditional $\gamma(\alpha)$ is also the minimizer of the average across N

$$\gamma(\alpha) = \arg\min_{\gamma\in\mathbb{R}^{d+1}} \mathbb{E}\left[\rho_{\Psi(\alpha|N)} (W - X\beta)\right],$$

which suggests the estimator

$$\widehat{\gamma}(\alpha) = \arg\min_{\gamma\in\mathbb{R}^{d+1}} \widehat{Q}(\gamma|\alpha) \text{ where } \widehat{Q}(\gamma|\alpha) = \frac{1}{L}\sum_{\ell=1}^{L} \rho_{\Psi(\alpha|N_\ell)} (W_\ell - X_\ell\gamma). \tag{3.11}$$

The private value conditional quantile can be estimated via $\widehat{V}(\cdot|X) = X\widehat{\gamma}(\cdot)$ and then used into equation (2.9) to estimate the seller's expected payoff $\Pi(\cdot|X, N, V_0)$. Although equation (2.10) in Proposition 2 gives a closed form to estimate the optimal reservation price $R_*(X, V_0)$, it involves an estimation of the quantile density function $V^{(1)}(\cdot|X)$. It is, therefore, more convenient to estimate $R_*(X, V_0)$ as in Li, Perrigne and Vuong (2003) based upon a maximization of the estimated expected payoff. This approach differs from Li et al. (2003) in the sense that the expected payoff depends upon the quantile level and for this reason it involves first an estimation of the optimal screening level.

The estimated expected payoff, the estimated optimal screening level and reserve price are

$$\widehat{\Pi}\left(\alpha|X,N,V_0\right) = V_0\left(X\right)\alpha^N + X'\widehat{\gamma}_{\tau_L}\left(\alpha\right)N\alpha^{N-1}\left(1-\alpha\right) + N\left(N-1\right)\int_{\alpha}^{1} t^{N-2}\left(1-t\right)X'\widehat{\gamma}_{\tau_L}dt,$$

$$\begin{aligned}
\widehat{\alpha}_R\left(X,V_0\right) &= \arg\max_{\alpha\in[0,1]} \widehat{\Pi}\left(\alpha|X,N,V_0\right), \\
\widehat{R}\left(X,V_0\right) &= X'\widehat{\gamma}_{\tau_L}\left(\widehat{\alpha}_R\left(X,V_0\right)\right).
\end{aligned}$$

where τ_L in $(0,1)$ is a sequence that goes to 0 when L grows and $\widehat{\gamma}_\tau\left(\cdot\right)$ is

$$\widehat{\gamma}_\tau\left(\alpha\right) = \begin{cases} \widehat{\gamma}\left(\tau\right) & \text{for } \alpha \text{ in } \left[0,\tau\right), \\ \widehat{\gamma}\left(\alpha\right) & \text{for } \alpha \text{ in } \left[\tau,1-\tau\right], \\ \widehat{\gamma}\left(1-\tau\right) & \text{for } \alpha \text{ in } \left(1-\tau,1\right]. \end{cases}$$

This redefinition of the estimated quantile regression slope ensures existence for extreme quantile levels near 0 and 1, provided τ_L is going to 0 slowly enough. Indeed, as seen from Bassett and Koenker (1982, fig. 2), the standard estimator of the extreme quantile regression slope coefficients are not unique and selecting some may give inconsistent estimators. An standard solution is to define them as limits when quantile levels go to 0 or 1 as performed in Bassett (1988) and studied in Chernozhukov (2005). The next Proposition gives conditions for consistency of this estimation approach.

Proposition 3 *Suppose that Assumptions 1-5 hold and that the population optimal screening level is unique. Then, if τ_L goes to 0 slowly enough, $\widehat{\alpha}_R\left(X,V_0\right)$ and $\widehat{R}\left(X,V_0\right)$ are consistent estimators of the optimal screening level and reserve price.*

Following Chernozhukov (2005), it is expected that Proposition 3 applies as soon as $L\tau_L$ is bounded away from infinity or diverges. The proof of Proposition 3 is given in the supplemental material.

3.1 Asymptotic Properties of the Estimator

In this section, the asymptotic properties[4] of the private value quantile regression estimator (3.11) are studied. In what follows, let $Q(\gamma|\alpha) = \mathbb{E}\left[\widehat{Q}(\gamma|\alpha)\right]$ and

$$\widehat{Q}(\widehat{\gamma}|\alpha) = \widehat{Q}(\widehat{\gamma}(\alpha)|\alpha) = \min_{\gamma \in \mathbb{R}^{d+1}} \widehat{Q}(\gamma|\alpha)$$

be the population and the optimized quantile regression objective functions. The first and second derivatives of $Q(\gamma|\alpha)$ with respect to γ will be denoted, respectively, by $Q_\gamma(\gamma|\alpha)$ and $Q_{\gamma\gamma}(\gamma|\alpha)$. Proposition 4 gives the asymptotic distribution of the private value quantile regression estimator (3.11).

Proposition 4 *Under assumptions 1-5,*

$$\sqrt{L}\left(\widehat{\gamma}(\alpha) - \gamma(\alpha)\right) \xrightarrow{d} \mathcal{N}\left(0, Q_{\gamma\gamma}^{-1}(\gamma|\alpha)\, \mathbb{E}\left[\Psi(\alpha|N)\left(1 - \Psi(\alpha|N)\right) X'X\right] Q_{\gamma\gamma}^{-1}(\gamma|\alpha)\right),$$

where

$$Q_{\gamma\gamma}(\gamma|\alpha) = \mathbb{E}\left[f_W\left(X\gamma(\alpha)|X,N\right) X'X\right],$$

$F_W(\cdot|X,N) = \Psi\left(F_V(\cdot|X)|N\right)$ *and* $f_W(\cdot|X,N)$ *being the c.d.f. and p.d.f. of the winning bids given* (X,N)*.*

The asymptotic variance of the quantile regression estimator can be estimated using tech-

[4]The supplemental material analyses the estimator performance in finite samples in comparison with a nonparametric estimation approach.

niques described in Koenker (2005). The applications considered here uses bootstrap inference, which does not require an estimation of the variance (see the supplemental material for more details)[5].

4 Exclusion Participation Restriction

Although standard in many econometric works as Guerre, Perrigne and Vuong (2000) among others, conditioning the private value distribution function on N is not usual in theoretical auction models, see e.g. Krishna (2010). This choice can be however motivated by unobserved heterogeneity or endogenous entry, as discussed below. Aradillas-Lopez et al. (2013) similarly interpret discrepancies across auctions with different number of bidders as resulting from unobserved heterogeneity or endogenous entry.

The first setting of interest is unobserved heterogeneity. Suppose that instead of X, the bidders observe an auction characteristic (X, X_u) that includes a component X_u not observed by the analyst. Hence, the private value quantile relevant for policy analysis is $V(\alpha|X, X_u)$, which cannot be estimated without further assumption. It can be for instance assumed that the actual number of bidders depends upon the auction characteristic, that is $N = N(X, X_u)$, in a way that fully captures the impact of the unobserved characteristic, i.e. $V(\cdot|X, X_u) = V(\cdot|X, N(X, X_u))$ such that the conditional quantile $V(\cdot|X, N)$ is fully relevant for policy analysis purposes.

A second motivation is given by the recent econometric literature on endogenous entry, see Gentry and Li (2014), Li and Zheng (2009) and Marmer et al. (2013). These models consider a two stage game, where the first stage is entry and the second stage is the auction

[5] The supplemental material extends Proposition 3 to a possibly misspecified nonlinear quantile regression model.

game. The structural parameter is the joint distribution of the private values and signals given the characteristic X, which is used in the entry stage of the game. The second stage involves an actual number of bidders N, who have decided to participate in the auction, and the conditional quantile $V(\cdot|X,N)$ of private values given (X,N). A key contribution of the aforementioned econometric literature is that the structural parameter is identified from N and $V(\cdot|X,N)$, so that estimation of the model can be performed through estimation of the conditional c.d.f. or quantile of private values given (X,N).

Therefore, the importance of a entry stage or potential unobserved heterogeneity affecting bidders' participation can be tested investigating the independence of private values and number of bidders. Note that a possible implication of the dependence upon N is that the optimal reservation price also depends upon the number of actual bidders N, implying that policy analysis based on a misspecified model may not be reliable.

Consider the case in which the true quantile is $V(\cdot|X,N)$. By assumption 3, $V(\cdot|X,N) = X\gamma(\alpha|N)$, and the coefficient $\gamma_n(\alpha) = \gamma(\alpha|N=n)$, for each $n \in \mathcal{I}$, is the parameter of interest. Let

$$\gamma_n(\alpha) = \arg\min_{\gamma \in \mathbb{R}^{d+1}} Q(\gamma|\alpha,n) \text{ where } Q(\gamma|\alpha,n) = \mathbb{E}\left[\rho_{\Psi(\alpha|N)}(W - X\gamma)\,|N=n\right].$$

An interesting way to investigate the exclusion participation restriction would be to consider the null hypothesis of independence, i.e. $V(\cdot|X) = V(\cdot|X,N)$, implying that

$$\begin{aligned} H_0 &: \gamma_n(\alpha) = \gamma(\alpha) \text{ for all } \alpha \in \mathcal{A} \text{ and } n \in \mathcal{I} \\ H_1 &: \text{not } H_0. \end{aligned}$$

A simple way of testing the null hypothesis above is via a Wald test, jointly for all the coefficients and quantile levels. The estimator $\widehat{\gamma}_n(\alpha)$ is the minimizer

$$\widehat{\gamma}_n(\alpha) = \arg\min_{\gamma\in\mathbb{R}^{d+1}} \widehat{Q}(\gamma|\alpha,n), \text{ where } \widehat{Q}(\gamma|\alpha,n) = \frac{1}{L_n}\sum_{\ell=1}^{L}\mathbb{I}(N_\ell = n)\,\rho_{\Psi(\alpha|N_\ell)}(W_\ell - X_\ell\gamma),$$

for L_n a subsample of auctions with $N = n$ bidders. However, such a test would involve standardization of the test-statistic by the variance-covariance matrix of the coefficients, which in turn involves estimation of the unknown density function of the random errors. The estimation of the latter requires either a bandwidth choice (see Powell (1991)) or bootstrap resampling methods (see Buchinsky (1995)). Some preliminary experiments had suggested that an alternative strategy as described below may give better results.

A strategy similar to a maximum likelihood ratio test can be implemented to avoid the estimation of the variance-covariance matrix. Let $\widehat{Q}(\widehat{\gamma}_n|\alpha,n)$ represents the optimized individual objective function. Under the null hypothesis, for $\underline{n},\cdots,\overline{n}\in\mathcal{I}$, $Q(\gamma|\alpha) = \sum_{n=\underline{n}}^{\overline{n}} Q(\gamma|\alpha,N=n)\,\mathbb{P}(N=n)$. This leads to consider the test statistic

$$M_{\text{Ind}} = 2L\left[\widehat{Q}(\widehat{\gamma}|\alpha) - \sum_{n=\underline{n}}^{\overline{n}}\widehat{Q}(\widehat{\gamma}_n|\alpha,n)\,L_n/L\right]. \tag{4.12}$$

The limiting distribution of (4.12) is studied in the supplemental material. To compute the critical values and p-values of test and avoid the estimation of the density function, I consider the random weighting bootstrap method proposed by Rao and Zhao (1992), Wang and Zhou (2004) and Zhao, Wu and Yang (2007).

5 Empirical Application

This section illustrates empirically the methodology using data from ascending timber auctions run by the USFS. Timber auctions data have been used in several empirical studies, see e.g. Baldwin, Marshall and Richard (1997), Haile (2001), Athey and Levin (2001), Athey, Levin and Seira (2011), Li and Zheng (2012), Aradillas-Lopez, Gandhi and Quint (2013), Li and Perrigne (2003) and others. Some other works have investigated risk-aversion on timber auctions, as e.g. Lu and Perrigne (2008), Athey and Levin (2001) and Campo, Guerre, Perrigne and Vuong (2011).

The dataset used here is publicly available on the internet[6] and aggregates ascending auctions from the states covering the western half of the US (regions 1-6 as labelled by the USFS) occurred in 1979. It contains 472 auctions (i.e. 472 winning bids) involving a total of 1175 bids and a set of variables characterizing each timber tract including the estimated volume of the timber measured in thousand of board feet (or mbf) and its estimated appraisal value given in Dollar per unit of volume. Hence, the vector of covariates Z_ℓ is two-dimensional grouping both the appraisal value per mbf and the volume of the timber[7].

The set of prescribed quantiles considered for estimation is $\mathcal{A} = \{0.12, 0.14, \ldots, 0.80\}$. A linear specification for the private value conditional quantile is assumed, although nonlinearity via an exponential specification has been also investigated[8]. The reservation price is announced prior to the auction and equals the appraisal value of the tract[9]. The first sealed bid auction stage used to qualify bidders for the ascending auction may select a certain number of bidders for the ascending auction stage, so that it might be interesting to test whether $V(\cdot|X, N)$ is relevant for the policy analysis to be conducted. Roberts and Sweeting (2013)

[6]The same dataset was used by Haile and Tamer (2003), Lu and Perrigne (2008) and Aradillas-Lopez et al. (2013), and it is available at the JAE Data Archieve website: http://qed.econ.queensu.ca/jae/2008-v23.7/lu-perrigne/

[7]The descriptive statistics of the dataset are given in the supplemental material.

[8]See the supplemental material for more details.

[9]It is well known that the reservation price is nonbinding. See e.g. Campo, Guerre, Perrigne and Vuong (2011), Haile and Tamer (2003) and Aradillas-Lopez at al. (2013).

found evidences of selective entry in California timber auctions due to entry cost for bidders conducting their own cruise. In the timber auctions used here bidders do not conduct their own cruise, therefore entry cost might be comparatively small. There are nevertheless other costs that may affect bidders participation, such as developing a market study, preparing the bids and attending to the auction. If indeed entry costs are not relevant for bidders' decision in participating, the optimal reservation price policy could be chosen independently of N as shown in Proposition 2-(ii). Table 1 below suggests that it is indeed the case here.

Table 1
Exclusion Participation Restriction

Null Hypothesis	M-Statistic	p-value
$\gamma_n(\alpha) = \gamma(\alpha)$ for all $\alpha \in \mathcal{A}$ and $n \in \mathcal{I}$	653.83	0.3096

Figure 1 and Table 2 describe the private values quantile regression coefficients and 95% confidence intervals for the estimates. The most interesting variable is the appraisal value, a quality measure released by the seller, which is often interpreted as the seller's private value[10]. The associated quantile regression slope coefficient is given in the third column of Table 2 and Figure 1a. Note that the coefficient is always significant and larger than 1, suggesting that it acts as a markup indicating how much more the auctioned good appraisal value is valued by the bidders than the seller. The private values can be also interpreted as a measure of how much the bidders would be willing to sell goods made with the timber bought at the auction[11]. This suggests that the higher the bidder's private value, the higher is his efficiency in aggregating value to the timber. The relative markup over the appraisal value

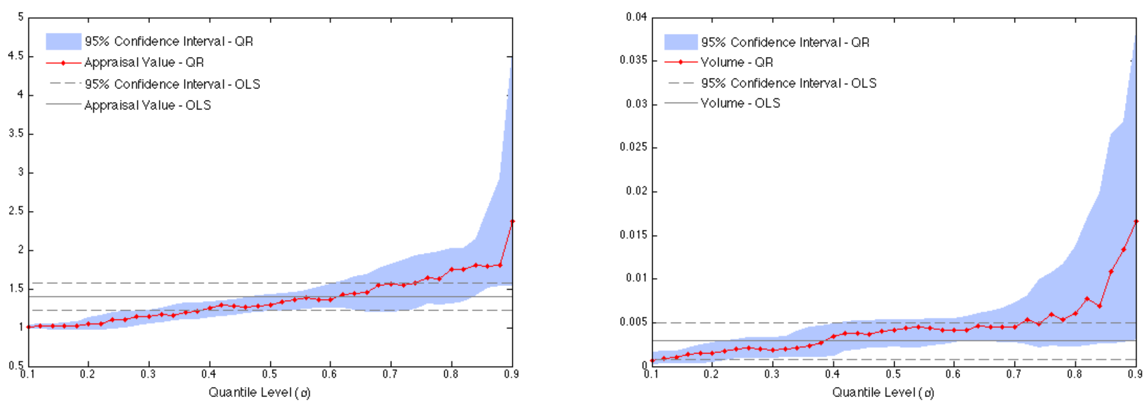
[10]See Lu and Perrigne (2008) and Aradillas-Lopez et al. (2013)

[11]In this interpretation, it is necessary to assume that timber is the most important component of the goods produced by the bidders. This could be however modified to cover other cases where timber would only be a part of these goods.

increases by about 75% when comparing bidders in the $\alpha = 0.10$ and $\alpha = 0.80$ quantiles of the private value conditional distribution. This is also evidence that bidders belonging to the upper tail of the private value distribution are more highly affected by changes in the appraisal value than median bidders. A test to investigate constancy of the slope coefficient has been applied showing significant change in the slope quantile regression coefficients across $\mathcal{A}$[12]. Figure 1a and 1b give, respectively, the quantile regression and OLS estimates of the appraisal value and volume with their corresponding 95% confidence intervals.

Figure 1: Slope Coefficients

(a) Appraisal Value (b) Volume



The 95% confidence intervals for the OLS estimate consider the heteroscedasticity-robust (White) standard errors. The ones for the quantile regression estimates were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair in each original subsample L_n.

[12]More details about the test procedure can be found in the supplemental material.

Table 2
Private Value Quantile Regression Estimates

Quantile Level	Intercept	Appraisal Value	Volume
0.1	0.95	1.01	0.0007
	[-0.97,2.28]	[0.99,1.04]	[0.0004,0.0016]
0.2	3.00	1.04	0.0016
	[-0.72,8.49]	[0.99,1.13]	[0.0005,0.0027]
0.3	9.39	1.15	0.0018
	[2.05,15.01]	[1.05,1.22]	[0.0010,0.0033]
0.4	11.77	1.25	0.0034
	[5.32,20.83]	[1.14,1.33]	[0.0013,0.0049]
0.5	21.03	1.29	0.0041
	[10.92,29.24]	[1.22,1.43]	[0.0023,0.0054]
0.6	35.68	1.36	0.0041
	[21.63,45.02]	[1.27,1.56]	[0.0029,0.0055]
0.7	44.28	1.57	0.0045
	[29.94,77.15]	[1.22,1.81]	[0.0029,0.0071]
0.8	67.64	1.75	0.0060
	[32.91,101.02]	[1.31,2.02]	[0.0024,0.0138]
0.9	72.98	2.37	0.0167
	[12.89,124.22]	[1.54,4.56]	[0.0031,0.0384]

The estimates are for a median auction. The 95% confidence interval in square brackets were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair in each original subsample L_n;

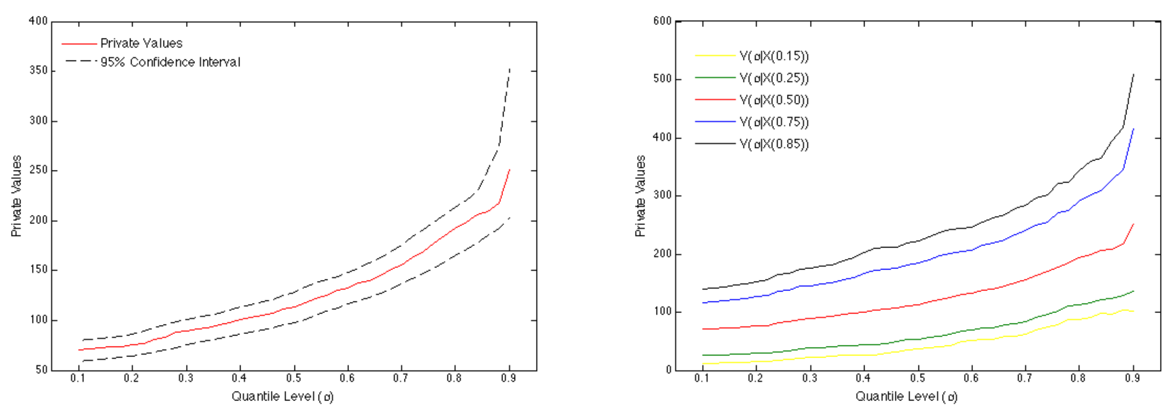
In what follows, $X_j(\tau)$ is the quantile of order τ of the random variable X_j, $j = 1, 2, 3$.

With some abuse of terminology, $X(\tau) = (X_1(\tau), X_2(\tau), X_3(\tau))$, for $X_1(\tau)$ a column vector of ones, denotes the quantile of order τ of the matrix of auction covariates X, i.e. $X(0.50)$ represents median auctions. Figure 2a gives the private value conditional quantile estimates for a median auction and their 95% confidence intervals. In a median auction, the volume is 967 thousand of board feet and the appraisal value is about \$68 per thousand of board feet. Figure 2b presents the quantile estimates for several quantile levels of $X(\tau)$, where $\tau = \{0.15, 0.25, 0.50, 0.75, 0.85\}$. In particular, it shows how the shape of the private value quantiles change due to variations in the quality and size of the timber tract. This effect becomes clearer when comparing a high with a low quantile of the private value conditional distribution. Consider in particular $\alpha = 0.12$ and $\alpha = 0.80$ for $X(\tau)$, $\tau = \{0.15, 0.50, 0.85\}$, that is, auctions with low, median and high quality and size. The relative increase in the private value is of about 600% in auctions with low quality and size, whereas it reduces to 172% and 142% in median and high quality and size auctions, respectively.

Figure 2: Private Values Conditional Quantiles

(a) Median Auction

(b) Auctions with Different Quality and Size



We now turn to the estimation of the seller's expected payoff and the associated optimal reservation price. The choice of the seller's private value $V_0(X)$ seems to be important to determine the optimal screening level policy since it represents the possible gains the seller may have when selling the good in the outside market. Note that $V_0(X) = 0$ may represent the case in which the seller has no opportunity to sell the good outside the auction. The most common choice for $V_0(X)$ is the appraisal value of the timber[13]. The results obtained for $V_0(X) = \text{Appraisal Value}\,(\text{AV})$ are compared with the case with no outside option $V_0(X) = 0$.

Table 3 gives the estimates of the optimal screening level $\widehat{\alpha}_R(X, V_0)$, the corresponding optimal reservation price $\widehat{R}(X, V_0)$ and the seller optimal expected payoff $\widehat{\Pi}(\widehat{\alpha}_R | X, N, V_0(X))$ for both choices of $V_0(X)$ and auctions with different quality and size. The optimal screening level $\widehat{\alpha}_R(X, V_0)$ is chosen as the maximizer of the seller's expected payoff[14] over $\mathcal{A}$. A general conclusion from Table 3 is that auctions with higher quality and size provide larger expected payoffs for the seller, whereas the optimal screening level reduces. A possible reason for that is the low heterogeneity among bidders observed in better auctions[15]. Therefore, as seen from Proposition 2.9, the seller has a strong incentive to use screening for low quality and size auctioned goods, whereas the seller private value's choice does not seem to affect his reservation price policy in these cases.

As mentioned in Section 2.1, the probability of trading in the auction with a screening level $\alpha_R(X)$ is $1 - \alpha_R(X)^N$. Table 4 groups the probabilities of trading under an optimal

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[13]As mentioned in Aradillas-Lopez et al. (2013), the seller's private value may be even lower than the appraisal value of the timber if exercising an outside option (through, for example, a lump-sum contract) entails additional cost to the seller. It is also possible that $V_0(X)$ is nevertheless higher than the appraisal value since scaled sales require the timber service to measure the timber actually harvested to calculate the payment. Therefore, by exercising the outside option the seller would avoid those costs.

[14]The definite integral in (2.9) is estimated via numerical integration using a trapezoidal rule over $\mathcal{A}$.

[15]Recall from Figure 2b that auctions with low quality and size show a significant increase in the markup over the timber appraisal value.

Table 3
Optimal Reservation Price

	Seller's Private Value	
	$V_0(X)=0$	$V_0(X)=\text{AV}$
$\widehat{\alpha}_R(X(0.15), V_0)$	0.75	0.75
	[0.52,0.80]	[0.56,0.80]
$\widehat{R}(X(0.15), V_0)$	83.08	83.08
	[38.93,104.58]	[45.34,109.03]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.15), 2, V_0)$	32.64	39.05
	[26.16,44.78]	[30.26,50.33]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.15), 3, V_0)$	38.98	43.80
	[30.62,52.92]	[33.32,56.73]
$\widehat{\alpha}_R(X(0.50), V_0)$	0.28	0.71
	[0.12,0.36]	[0.56,0.78]
$\widehat{R}(X(0.50), V_0)$	87.93	159.89
	[67.73,101.96]	[123.40,192.43]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.50), 2, V_0)$	91.71	108.49
	[79.17,102.32]	[95.11,122.13]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.50), 3, V_0)$	102.93	112.50
	[99,126.57]	[89.63,114.26]
$\widehat{\alpha}_R(X(0.85), V_0)$	0.24	0.56
	[0.12,0.42]	[0.47,0.80]
$\widehat{R}(X(0.85), V_0)$	163.65	243.97
	[135.51,213.63]	[220.03,376.25]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.85), 2, V_0)$	177.66	203.26
	[163.40,192.73]	[187.71,223.19]
$\widehat{\Pi}(\widehat{\alpha}_R \mid X(0.85), 3, V_0)$	196.37	208.61
	[179.89,212.72]	[191.63,231.26]

The 95% confidence intervals in square brackets were computed by resampling with replacement the (X_ℓ, W_ℓ)-pair in each original subsample L_n.

screening level policy according to $(X(\tau), V_0(X))$ and for $\tau = \{0.15, 0.50, 0.85\}$. Note that in auctions with low quality and size, the probability of selling the good is very low (44%

and 58% for $N = 2$ and $N = 3$, respectively). This is because bidders are very heterogeneous and the seller should set a high screening level to avoid low bidders from participating. This somehow carries over for median and higher quality and size auctions when the seller's private value is the appraisal value. Policy recommendations with such a low probability of selling may not make sense in practice, especially for goods with a potential high storage cost. This effect seems to be less expressive when the seller faces no outside option. As can be seen, the practical implementation of the auction theory can be sometimes difficult in the sense that usual choices for the seller's private value may lead to recommendation of mechanisms with very low probability of trading. This may question the relevance of considering expected payoff in the maximization process.

Table 4: Probability of Trading

		$X(0.15)$	$X(0.50)$	$X(0.85)$
$V_0(X) = 0$	$N = 2$	44%	92%	94%
	$N = 3$	58%	98%	99%
$V_0(X) = \text{AV}$	$N = 2$	44%	50%	69%
	$N = 3$	58%	65%	82%

6 Conclusion

This paper proposes an identification and estimation approach based on quantile regression to recover the bidders' private values conditional distribution. The quantile regression framework provides a flexible and convenient parametrization of the private value distribution, with an estimation methodology easy to implement and with various specification tests that can be derived. The paper shows that a focus on the quantile level of the private values distribution, in particular the optimal screening level, can be very useful for policy recommendation.

The empirical application using timber auctions from the USFS shows that policy recommendations should be carefully examined before practical implementation. The screening level associated with the optimal reservation price is fairly high in general, resulting in a low probability of trading. The analysis of the shape of the private value conditional quantile curves suggests that such inappropriate recommendations are due to a sharp increase in the private value conditional quantiles, which may be evidence of large heterogeneity among the bidders. As a consequence, the seller has a strong incentive to screen bidders' participation by using a high reservation price, leading then to a low probability of selling the auctioned good.

The estimated private values quantile shapes can be genuine but can also be the consequence of a model misspecification. It would be for instance interesting to investigate identification under a more flexible, and perhaps unknown, quantile specification function. Gimenes and Guerre (2014) have suggested a sieve interactive quantile specification in the setup of first-price sealed bid auctions, which can cope with dimension reduction issues while still very flexible and with nonparametric features.

Regarding the consequences of the quantile shapes for policy recommendations, some other works have also noticed such a high level of the optimal reservation price in timber auctions. Aradillas-Lopez et al. (2013) suggest that neglecting private values affiliation can generate high reservation prices. However, their nonparametric methodology may be affected by the curse of dimensionality. In addition, as noted in Roberts and Sweeting (2013), timber auctions include a preliminary selection that can affect the estimated shape of the private value quantile functions. The large heterogeneity revealed by the estimation of the private value conditional quantile function can also be an indication of asymmetry. As discussed in Cantillon (2008) and Gavious and Minchunk (2012), sellers facing asymmetry have an incentive to increase competition by increasing reservation prices.

However, analysing revenue with a risk neutral seller perspective may not be appropriate to address issues such as high reservation prices and low probability of selling the auctioned object. The results given in Hu, Matthews and Zou (2010) regarding risk aversion affecting sellers can be useful to provide more relevant reservation price recommendations. Gimenes (2014) proposes a numerical investigation of the variation in the optimal screening level when the seller has a constant relative risk aversion utility function and concludes that considering risk averse sellers is indeed sufficient to achieve reasonable policy recommendations.

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